HEXAFLEXAGON

Unit No. 236

Things of science

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Paper folding and doodling may be a time passer for many, but when a British mathematics student at Princeton University folded paper strips, he made an intriguing discovery. By folding a strip diagonally at three places and joining the ends to form a hexagon, the student discovered that by pinching two adjacent triangles together and pushing the opposite triangles of the hexagon toward the center the hexagon would open out showing completely new faces. Using longer strips of paper, models were constructed that had the same hexagonal shape but could be opened to six different faces. Since six triangles form each face of the figures and the structures have the ability to flex, the models were named hexaflexagons.

In this unit of THINGS of science you will find materials to help you in the construction and understanding of hexaflexagons: a roll of paper, three colored sheets of paper, two colored pencils and a protractor.

Since the basis of hexaflexagons is six equilateral triangles forming a hexagon, first let us make an equilateral triangle.

Experiment 1 -- In order to fold an equilateral triangle you must start with a square. Using one of the colored sheets of paper included in this unit, take one corner and bend in the direction of the diagonally opposite corner. Fold the sheet so that one of the short sides falls upon one of the long sides and their edges meet exactly. You must be very careful when making your folds or your figures will not be accurate. Now fold the triangles over the remaining portion of the sheet and using a knife, cut off this remaining portion. Unfold the sheet and you will find that you have a square.

Measure the angles with the protractor included in this unit, by placing the black dot of the protractor on the vertex of one of the angles with one side of the angle lying along the 0 degree line. Does the angle measure 90 degrees? You will find all four angles are equal as well as the sides.

With this square for a model fold the ends of the other two colored sheets over the square and cut off the remaining portion so that the two sheets are perfect squares containing no folds.

Experiment 2 -- Take one of your colored squares and fold it double, laying one opposite edge upon the other carefully. You will obtain a crease which passes through the mid-points of the remaining sides. Hold the square so that your crease is perpendicular to the lower side of the square. Starting at the left, label the ends of the lower side A and B. Turn the point A to the point on the crease so that the distance from point B to the point on the crease is equal to AB. Make a fold and mark the point on the crease C. The side AB is now indicating one side of your equilateral triangle. Fold along this side, which becomes BC of your triangle. Now make a fold along points A and C. Unfold your sheet and measure the sides of the triangle ABC. Are they equal? The sides of an equilateral triangle, as the name implies, are equal and the angles all measure 60 degrees. Check the size of the angles with your protractor.

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A hexagon is a six-sided figure; the prefix hexa, derived from Greek, means six.

Experiment 3 -- Using your square, take one corner and place it on the diagonally opposite corner, being careful to line up the edges of the square exactly. Crease the fold so that you have two triangles. Find the mid-point of the fold and mark it. Bend one of the 45 degree angles over the opposite side so that

the three folds at the mid-point make two angles of 60 degrees and the vertex of the angles fall at the mid-point.

Now fold the other 45 degree angle over to its opposite side so that one 60 degree angle is on top of the other. Turn your figure over; it should appear as a kite with two wings extending from the kite. Turn down the top portion of the kite, thus forming an equilateral triangle. Fold the wings over the base of the triangle.

Unfold your figure and you will find a regular hexagon, one in which all sides and angles of the six-sided figure are equal. Measure the sides of the hexagon and the angles. Are the sides equal and what do the angles measure? If your hexagon was folded accurately, your angles should all be 120 degrees. Can you identify the six figures that make up the hexagon? To determine what they are, measure their angles and sides. Now you see that a regular hexagon is made up of six equilateral triangles. This is the basis for the hexaflexagons.

CONSTRUCTING A TRIHEXAFLEXAGON

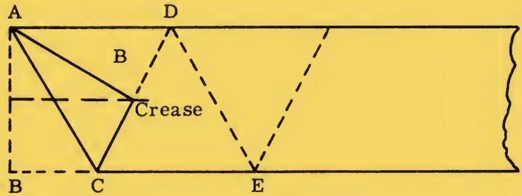
The simplest of the hexaflexagons is the trihexaflexagon which can flex to show three faces.

Experiment 4 -- Take the roll of paper included in this unit and fold the end over matching the edges carefully. Crease the fold well and with a knife cut off the end, thus obtaining a square end. Fold about two and one half inches of the paper tape in half lengthwise and crease the fold, being very careful to make the edges meet, because the accuracy of your further work depends on this crease. This produces the crease in Figure 1.

Then,as shown in Figure 1, fold the end AB over the rest of the tape so that the corner B lies accurately on the middle lengthwise crease and the fold ends accurately at the corner A. Crease the fold well. You have now located point C. If this is done accurately, the angle ACB will be as accurate a 60 degree

angle as could be produced with an ordinary compass. Confirm this with your protractor.

FIGURE 1



Take corner A and fold it through point C until the fold AC lies accurately on the lower edge of the tape. Crease the fold well. If this is done accurately the fold will lie under the edge of BC and locate the points D and E. The point D will form corners of the equilateral triangles ACD and CDE.

Now take the corner D and fold it forward about point E until the fold DC lies accurately on the upper edge of the tape. Crease the fold well. Repeat this process folding alternately forward and backward until a chain of some 14 equilateral triangles is formed. It is very important that in all this folding, the corners and edges be accurately matched.

Cut off the first two triangles to get rid of the lenghtwise crease which served merely to construct the 60 degree angle. Count of 10 (3 x 3 + 1) triangles and cut them off leaving two or three equilateral triangles to start another chain. Since the creases between the triangles serve as hinges in the completed flexagon, it is desirable to reverse all of the creases so the flexagon will operate easily.

<u>Experiment 5</u> -- Lay the chain of equilateral triangles flat and label both faces as shown in Figure 2. Make very light pencil marks when indicating the numbers for you will later wish to erase them. Fold each pair of adjacent 2's together, and your chain will fold into a hexagon. The tenth triangle overlaps the

first. Paste these triangles together by their unlettered sides 3 and 1. The trihexaflexagon will appear as a hexagon with the triangles on the top face all labeled 1 and those on the bottom labeled 3. Color the top of the flexagon labeled 1 with the blue pencil included in this unit and the bottom of the flexagon, labeled 3, red.

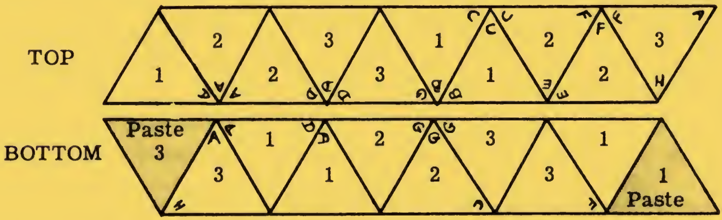


FIGURE 2

<u>Experiment 6</u> -- To operate the trihexaflexagon take up the figure with the blue face on top. With the left thumb and forefinger pinch the blue corner labeled AA, thus bringing the opposite red triangles labeled AA together. Push in the blue corner labeled BB. With the right thumb pull out the blue corner labeled CC. The hexaflexagon will then open up flat from above with the face labeled 2 on top. This face you will not color but leave white. The process of opening the flexagon to show a new face is called a flexion.

<u>Experiment 7</u> -- If on the white face the corner labeled AA is pinched bringing the opposite blue triangles labeled AA together (the point at which the flexagon was first flexed), and the white corner labeled CC is pushed in, the flexagon will not open on top. The flexion is blocked.

By turning the trihexaflexagon clockwise 60 degrees the white corner labeled DD will be between the left thumb and forefinger. Pinch it together thus bringing the opposite blue triangles labeled DD together. Push in the white corner labeled EE and

pull out the white corner labeled FF. The hexaflexagon will then open up flat from above with the red face showing.

If you attempt to bring up the blue face by flexing the flexagon on the red face at DD (the point just used to bring up the red face from the white face), you will find that this flexion is blocked. Again turn the flexagon clockwise 60 degrees to the red corner labeled GG. Pinch it together bringing the opposite white triangles labeled GG together. Push in the red corner labeled HH and pull out the red corner labeled AA. The hexaflexagon will then open up flat from above with the blue face on top. This is the original position of your trihexaflexagon.

The letters are just a guide to help you follow the sequence of flexing. After operating the trihexaflexagon for some time it will become obvious that upon flexing it about one corner an attempt to flex it again about the same corner will be blocked, but if it is rotated clockwise 60 degrees each time in sequence, the flexions can be repeated indefinitely. You will note that each of the flexions places the face previously on top of the trihexaflexagon on the bottom, and brings on top a face which before the flexion was buried in it. This is characteristic of all flexagons.

As you become familiar with and continue to flex your trihexaflexagon you may discover an easier way to bring up each successive face. This can be done by folding the corner you wish to use as your starting point and bringing the opposite corner flat against the triangles that are held together and the flexagon will open from its apex.

Experiment 8 -- With the blue face of your trihexaflexagon on top draw a line from the apex of each equilateral triangle so that it bisects the base of each triangle. You now have six lines radiating from the center of the hexagon which divide in half each of its six sides. Now bring up the white face of your trihexaflexagon, and draw lines in the same manner radiating from the center of the hexagon. Turn your flexagon over and look at the blue face.

How are the lines that you drew radiating from the center now arranged? Turn your flexagon back to the white face and bring up the red face. Draw the six lines from the center of the figure. Are the lines on the white face arranged in a triangle?

As you continue to flex your trihexaflexagon you will always find the lines radiating from the center on one face and a triangle formed on the other face.

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To bring up all the faces of any hexaflexagon, continue to flex it about one corner until it is blocked. If it is blocked, never try to force it, but shift to the next corner in rotation. It will open here at least once. Flex it about this corner until it is blocked, then shift to the next corner in sequence. Continue this process until the original face reappears. This process will bring to the top all of the faces of the hexaflexagon, some of them several times, but all of them at least once. A complete cycle is called a traverse.

Experiment 9 -- A traverse is best illustrated by the simplest of the hexahexaflexagons, the straight chain hexahexaflexagon, which has six faces. To construct a straight chain hexahexaflexagon, lay off on the paper tape a chain of 19 (3 x 6 + 1) equilateral triangles as you did in Experiment 4 and label the tri-

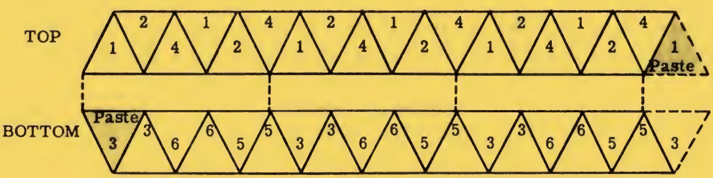

FIGURE 3

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angles on the two faces as shown in Figure 3.

In folding, we first fold up each third into a triangular stack, keeping the end triangles (with faces labeled 1 and 3) free for pasting. Fold the first third together by folding in turn the triangles labeled 6 to face each other, 5 to 5, 4 to 4, and 2 to 2. Hold this stack together with a paper clip. Repeat for the second third and hold together with a paper clip. Upon folding the final third, you will find that a hexagon is formed with the first and the last triangles overlapping. Paste them together, forming the hexahexaflexagon labeled 1 on one face and 3 on the other.

Starting with the 1 on top and 3 on the bottom, the hexahexaflexagon can be flexed three times about the same corner before it is blocked. Turning it 60 degrees it can be flexed only once about that corner before it is blocked. About the next corner it can again be flexed three times.

A traverse, or complete cycle, of this hexahexaflexagon thus consists of a sequence of 3, 1, 3, 1, 3, 1 flexions in order. After this sequence has been completed, it can be repeated indefinitely. Each cycle consists of 12 flexions. Starting at any condition a traverse of 12 flexions will bring the hexahexaflexagon back to its original condition. In the particular traverse chosen the faces will appear in order on the top and bottom as follows:

TOP:	1	2 4 5	2	4 1 6	4	1 2 3	1	2 4 5	2	etc.
BOTTOM:	3	1 2 4	5	2 4 1	6	4 1 2	3	1 2 4	5	etc.

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